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# Proposed Test for a Hidden Variables Theory

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A hidden variables model for quantum mechanics is proposed and a possible test for its validity is described.

## 1. INTRODUCTION

Starting with the quantum logic approach to quantum mechanics we assume the existence of an order-determining set of dispersion-free states and obtain a hidden variables theory. This theory has an important feature which is not present in the traditional Hilbert space framework of quantum mechanics, namely, the expectation functional is not necessarily additive. We propose this as a possible test for the existence of hidden variables.

## 2. HIDDEN VARIABLES MODEL

The quantum logic approach has proved useful for investigating the theoretical structure of quantum systems (Birkhoff and von Neumann, 1936; Jauch, 1968; Mackey, 1963; Piron, 1976; Varadarajan, 1968). Although there are variations to this approach one usually obtains at least a $\sigma$-orthocomplete orthomodular poset. We first recall the relevant definitions.

Let $L$ be a set of elements called *propositions*. The propositions correspond to yes–no experiments for a quantum system. We assume that $L$ is a partially ordered set (poset) with first and last elements $0, 1$, respectively. We also postulate the existence of an orthocomplementation $': L \to L$ satisfying : $a'' = a$, $a \leqslant b$ implies $b' \leqslant a'$, and $a \vee a' = 1$ for all $a, b \in$

$L$. We say that $a,b \in L$ are *orthogonal* ($a \perp b$) if $a \leqslant b'$ and if $a_i \in L$ are mutually orthogonal we assume that $\bigvee a_i$ exists in $L$. Finally, if $a \leqslant b$ we assume that $b = a \bigvee (b \bigwedge a')$. The resulting structure is a $\sigma$-orthocomplete orthomodular poset or *logic*.

There are two common examples of logics. Let $H$ be a complex Hilbert space and let $L(H)$ be the set of all orthogonal projections on $H$. Under the usual order and orthocomplementation, $L(H)$ becomes a logic. This is the logic of traditional quantum mechanics. Let $\Omega$ be a nonempty set and let $L(\Omega)$ be a $\sigma$-algebra of subsets of $\Omega$. Under set-theoretic inclusion and complementation, $L(\Omega)$ becomes a logic corresponding to classical mechanics. As we shall see, there is another distinct alternative which gives a hidden variables model for quantum mechanics.

A *state* on $L$ is a map $m: L \to [0,1]$ satisfying $m(1) = 1$ and $m(\bigvee a_i) = \Sigma m(a_i)$ for any sequence of mutually orthogonal elements $a_i \in L$. We interpret $m(a)$ as the probability that the result of a measurement of the proposition $a$ is "yes" when the system is prepared in the state $m$. A state $m$ is *dispersion free* if its only values are 0 and 1. A dispersion-free state provides a deterministic description since every proposition has either a "yes" or "no" answer with certainty when the system is prepared in such a state. A set of states $S$ is *order determining* if $m(a) \leqslant m(b)$ for every $m \in S$ implies that $a \leqslant b$. The traditional Hilbertian logic $L(H)$, $\dim H > 2$, has no dispersion-free states (Gudder, 1973). However, a classical logic $L(\Omega)$ has an order-determining set of dispersion-free states. Indeed, the set of probability measures concentrated at the points of $\Omega$ provides such a set.

We say that a quantum logic $L$ *admits hidden variables* if $L$ has an order-determining set of dispersion-free states. The important point is that there are logics other than the classical ones which admit hidden variables and these retain the interference effects characteristic of quantum mechanics.

We say that two propositions $a, b \in L$ are *compatible* ($a \leftrightarrow b$) (Mackey, 1963; Varadarajan, 1968) if there exist mutually disjoint propositions $a_1$, $b_1$, and $c$ such that $a = a_1 \bigvee c$ and $b = b_1 \bigvee c$. In $L(\Omega)$ any two propositions are compatible, while in $L(H)$ two propositions are compatible if and only if they commute. The existence of noncompatible elements is probably the most important distinguishing feature of quantum mechanics as compared to classical mechanics. Noncompatibility accounts for the interference of measurements that frequently occurs in quantum mechanics as exemplified by the Heisenberg uncertainty principle.

We now present a class of logics which admits hidden variables. Let $\Omega$ be a nonempty set. A collection of subsets $C$ of $\Omega$ is called a *$\sigma$-class* if (1) $\Omega \in C$, (2) $A \in C$ implies that the complement $A^c \in C$, and (3) $\cup A_i \in C$ for any sequence of mutually disjoint $A_i \in C$. It is not hard to check that a $\sigma$-class under the usual set-theoretic order and complementation is a logic.

Moreover, for $A, B \in C$, we have $A \leftrightarrow B$ if and only if $A \cap B \in C$. The set of probability measures concentrated at the points of $\Omega$ form an order-determining set of dispersion-free states so $C$ admits hidden variables. Clearly, a $\sigma$-class is a generalization of the $\sigma$-algebra for a classical logic.

We now give an example of a $\sigma$-class which might have physical importance. Let $\lambda > 0$ and let $\Omega = [0, n\lambda]$ where $n > 1$ is an integer. If $C$ is the collection of Lebesgue measurable subsets of $\Omega$ whose Lebesgue measures $\mu$ are integer multiples of $\lambda$, then $C$ is a $\sigma$-class but not a $\sigma$-algebra. This example can be used as a basis for an "elementary length" theory in which $\lambda$ is the elementary length (Gudder, 1968, Gudder and Marchand, 1980). Since there are many incompatible elements in $C$, this example exhibits the interference effects characteristic of quantum mechanics.

We have seen that a $\sigma$-class admits hidden variables. Conversely, suppose a logic $L$ admits hidden variables and let $S$ be an order-determining set of dispersion-free states on $L$. Let $h$ be the map from $L$ into the collection of subsets of $S$ defined by $h(a) = \{m \in S : m(a) = 1\}$ and let $C$ be the range of $h$. It can be shown that $C$ is a $\sigma$-class in $S$ and that $h : L \to C$ is an isomorphism (Gudder, 1973). We thus have the following result.

> *Theorem.* A logic $L$ admits hidden variables if and only if $L$ is isomorphic to a $\sigma$-class $C$. In this case $C$ can be taken to be a $\sigma$-class of subsets of the set of dispersion-free states on $L$.

How does this fit in with the nonexistence proofs for hidden variables? Most of these proofs show that if a model admits hidden variables then the model must be classical in the sense that all elements of the model are mutually compatible. This is not the case in the above theorem. The reason for the difference is that in the nonexistence proofs more restrictions are placed in the definition of a state than we have made. For example, in Jauch and Piron (1963) it is assumed that a state $m$ satisfies the following condition: if $m(a) = 1$ and $m(b) = 1$, then $m(a \wedge b) = 1$ [$L$ is assumed to be a lattice in Jauch and Piron (1963)]. This condition has been criticized in many sources (for example, Bohm and Bub, 1966).

## 3. NONADDITIVITY OF EXPECTATIONS

We now consider a possible test for hidden variables in quantum mechanics. Since it is probably impossible to construct a dispersion-free state in the laboratory one cannot get a direct test. However, there is another important difference between logics that admit hidden variables and traditional Hilbertian logics which suggests an indirect test. The difference is that in the hidden variables theory expectations need not be additive.

An *observable* on a logic $L$ is a $\sigma$-homomorphism from the Borel subsets of the real line $B(\mathbb{R})$ into $L$. Applying the spectral theorem, an observable on a Hilbertian logic $L(H)$ can be identified with a self-adjoint operator on $H$. Moreover, by Gleason's theorem (Gleason, 1957), any state $m$ on $L(H)$ ($\dim H > 2$) has the form $m(P) = \operatorname{tr}(WP)$, $P \in L(H)$, where $W$ is a density operator. The *expectation* $E_m$ of an observable (self-adjoint operator) $A$ then becomes $E_m(A) = \operatorname{tr}(WA)$. If $A$ and $B$ are self-adjoint operators, then

$$E_m(A+B) = \operatorname{tr}[W(A+B)] = \operatorname{tr}(WA) + \operatorname{tr}(WB) = E_m(A) + E_m(B)$$

so the expectation functional is additive.

Now let $C$ be a $\sigma$-class of subsets of $\Omega$. A function $f:\Omega \to \mathbb{R}$ is *measurable* if $f^{-1}(E) \in C$ for every $E \in B(\mathbb{R})$. If $x: B(\mathbb{R}) \to C$ is an observable, there exists a unique measurable function $f$ such that $x(E) = f^{-1}(E)$ for all $E \in B(\mathbb{R})$, and conversely if $f:\Omega \to \mathbb{R}$ is measurable, then $f^{-1}$ is an observable. Thus, observables on $C$ can be identified with measurable functions. Let $\mu$ be a state (probability measure) on $C$ and let $f$ be a measurable function on $\Omega$. Since $C_f = \{f^{-1}(E) : E \in B(\mathbb{R})\}$ is a $\sigma$-algebra, $(\Omega, C_f, \mu)$ is a probability space and we define the expectation $E_\mu(f) = \int f\,d\mu$ in the usual way. The following example, due to Zerbe (1979), shows that the expectation functional need not be additive.

Let $Z$ denote the integers and let $\Omega = \{(i,j) : i,j \in Z\}$. Let $R_m = \{(i,j) : i = m\}$, $C_n = \{(i,j) : j = n\}$, and $D_t = \{(i,j) : i + j = t\}$. Let $\Sigma_R, \Sigma_C, \Sigma_D$ be the $\sigma$-algebras generated by $\{R_m : m \in Z\}, \{C_n : n \in Z\}, \{D_t : t \in Z\}$, respectively. Since $R_m \cap C_n \neq \varnothing$, $R_m \cap D_t \neq \varnothing$, $C_n \cap D_t \neq \varnothing$ for all $m, n, t$, it follows that $C = \Sigma_R \cup \Sigma_C \cup \Sigma_D$ is a $\sigma$-class. Define the probability measure $\mu$ on $\Sigma$ by $\mu(R_0) = \mu(C_0) = \mu(D_1) = 1$, $\mu(R_m) = \mu(C_n) = \mu(D_t) = 0$ for $m, n \neq 0$, $t \neq 1$. Define the measurable functions $f, g$ by $f(i,j) = i$, $g(i,j) = j$. Then $(f+g)(i,j) = i + j$ is measurable. Now $E_\mu(f) = E_\mu(g) = 0$ but $E_\mu(f+g) = 1$.

This suggests the following test for hidden variables. Find two noncompatible observables $A$ and $B$ such that $A$, $B$ and $A+B$ can be measured in the laboratory. Prepare a state $m$ and compute $E_m(A)$, $E_m(B)$, and $E_m(A+B)$. If $E_m(A+B) \neq E_m(A) + E_m(B)$ within the allowable experimental error, then there is a strong indication of the existence of hidden variables. Admittedly, such an experiment might be extremely difficult to perform. One possibility that comes to mind is to let $A$ and $B$ be the kinetic and potential energies of a system, respectively. Then $A+B$ would be the total energy.

## 4. TWO-DIMENSIONAL EXAMPLE

We have already noted that if $\dim H > 2$, then $L(H)$ has no dispersion-free states and hence does not admit hidden variables. We shall now show that if $\dim H = 2$, then $L(H)$ admits hidden variables and the expectation functional need not be additive. Since such logics are used to describe spin-$\frac{1}{2}$ systems, this gives a physically applicable Hilbertian logic that admits hidden variables.

Let $H$ be the two-dimensional Hilbert space $\mathbb{C}^2$. The nontrivial projections in $L(H)$ are one dimensional. Given two different one-dimensional projections $a, b \in L(H)$, a dispersion-free state $m$ can be constructed satisfying $m(a) = m(b') = 1$, $m(a') = m(b) = 0$. It follows that $L(H)$ admits hidden variables.

Let $a, b$ be the following one-dimensional projections:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and let $A, B$ be the spin observables

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a - a', \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = b - b'$$

The sum

$$A + B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

has eigenvalues $\pm\sqrt{2}$ and hence has the form $A + B = \sqrt{2}\, d - \sqrt{2}\, d'$, where $d, d'$ are one-dimensional projections. Since $A + B$ does not commute with $A$ or $B$, all the above one-dimensional projections are distinct. Define a state $m$ on $L(H)$ by $m(0) = 0$, $m(I) = 1$, $m(a) = 1$, $m(a') = 0$, and $m(c) = \frac{1}{2}$ for every $c \neq 0, 1, a, a'$. The expectations become

$$E_m(A) = m(a) - m(a') = 1$$

$$E_m(B) = m(b) - m(b') = 0$$

$$E_m(A + B) = \sqrt{2}\, m(d) - \sqrt{2}\, m(d') = 0$$

Hence, $E_m(A + B) \neq E_m(A) + E_m(B)$ and $E_m$ is not additive.

It is granted that dispersion-free states and the state $m$ defined above are not physical in the sense that they cannot be prepared in the laboratory. However, it could be argued that such states correspond to hidden variables which exist but are presently inaccessible. It should also be noted that the sum here is different than the sum of the corresponding measurable functions on a $\sigma$-class.

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